Sankhyā : The Indian Journal of Statistics
2018, Volume 80-B, Part 1, pp. 1-18

Application of Two Gamma Distributions Mixture to Financial Auditing

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Abstract

The considered problem can be treated as a particular topic in the field of testing some substantive hypothesis in financial auditing. The main theme of the paper is the well-known problem of testing hypothesis on admissibility of the population total of accounting errors amounts. The set of items with non-zero errors amounts is the domain in the accounting population. The book amounts are treated as values of a random variable which distribution is a mixture of the distributions of correct amount and the distribution of the true amount contaminated by error. The mixing coefficient is equal to the proportion of the items with non-zero errors amounts. The mixture of two gamma distributions is taken into account. The well-known method of moments and likelihood method are proposed to estimate parameters of distribution. It let us construct some statistic to test the outlined hypothesis. Moreover, the well-known likelihood ratio test is considered.

AMS (2000) *subject classification.* Primary: 62H15; Secondary: 91B28.
Keywords and phrases. Statistical auditing, Accounting error, Mixture of probability distribution, Method of moments, Likelihood ratio test

1 Introduction

Auditing costs of controlled accounting systems can be reduced due to usage statistical methods. Due to this, the number of applications of statistics in auditing is gradually increasing. Moreover, it is expected that statistical methods can also help to prepare audit reports as quickly as possible. There are many monographs and papers about statistical methods useful in auditing (see, e.g., Guy and Carmichael (1986), Marazzi and Tillé (2016), Särndal et al. (1989), Tamura (1988)). Although the already large amount of statistical literature auditing is constantly expanding, it seems that many problems still need more efficient solutions.

Usually, statistical inference considered in auditing is focused on estimation of the total or the mean error amount. The inference is based on data observed in samples drawn from a population of documents. An audit sample delivers two pieces of information: the book (recorded) amount and the audited (correct or true) amount. Let us note that a value of the book amount can be a sum of smaller book amounts. The book values will be treated as observation of a random variable distributed according to mixture of two probability distribution. One of them is the distribution of audited amounts and the second one is the distribution of audited amounts contaminated with errors which not necessary should be additive to the correct amounts. For instance, the non-correct amount could be a non-linear function of errors and true amounts. But in practice, it is convenient when the error amount is observed as a difference between the non-correct and true amounts.

Inference on the total error amount is usually based confidence intervals. Of course, they are related to testing problems. Below, the decision-making process in auditing is treated as a problem of testing statistical hypotheses about admissibility of the total or the mean accounting errors. This approach lets us control not only significance level (risk of incorrect rejection) but also probability of appearing the type II error (risk of incorrect acceptance).

We can expect that book amounts distribution is positively skewed because small book amounts are more frequent than large book amounts. Moreover, it means that correlation coefficient between the sample mean and variance of book amounts is proportional to positive value of the third central moment of the book amounts distribution (see, e.g., Cramér (1962) or Frost and Tamura (1986)). Hence, that is why distributions of accounting amounts, expenditures, or income are modeled, e.g., by means of gamma, Poisson, or log-normal probability distributions.

Our consideration are based on the (superpopulation) model approach which is similar to that introduced, e.g., by Cox and Snell (1979). Let us note that its simplified version called fixed and finite population (design) approach was taken into account, e.g., by Fienberg et al. (1977).

Basic notation and definition are introduced in the next section. The considered model of accounting data is defined as mixture of two probability distribution. The hypothesis on the mean accounting error is formulated. This hypothesis is tested under the assumption that the accounting data are generated according to mixture of two gamma distributions in Section 3. Several test statistics are formulated based on estimators of gamma distribution parameters evaluated by means of the method of moments in Section 3.2. The ratio likelihood test is constructed in the next subsection. Monte Carlo method is adapted to approximation of probability distribution of the test statistic in Section 3.4. Some possible directions of developing the considered methods are discussed in the conclusion of this paper.

2 Model of Accounting Observations

Let U be the population of accounting documents of size N where accounting amounts are observed. Some of them are contaminated by errors. In the population U, there are book amounts x_i observed for each population element, $i \in U$. Let $\boldsymbol{x}^T = [x_1 x_2 ... x_N] \in R_+^N$, be the observation of the random vector $\boldsymbol{X}^T = [X_1 X_2 ... X_N]$. The true (without errors) accounting amounts are denoted by the values y_i, $i \in U$, and let $\boldsymbol{y} = [y_1 ... y_N] \in R_+^N$, be the outcome of the random vector $\boldsymbol{Y}^T = [Y_1 ... Y_N]$. The vector of accounting amounts contaminated by errors will be denoted by $\boldsymbol{w}^T = [w_1 ... w_N]$ and $\boldsymbol{w}$ is a value of $\boldsymbol{W}^T = [W_1 ... W_N]$. Finally, let $\boldsymbol{Z}^T = [Z_1 ... Z_N]$ where $Z_i = 0$ ($Z_i = 1$) if and only if $X_i = Y_i$ ($X_i = W_i$), $i \in U$. We assume that the rows $[X_i\ Y_i\ W_i\ Z_i]$, $i \in U$, of matrix $[\boldsymbol{X}\ \boldsymbol{Y}\ \boldsymbol{W}\ \boldsymbol{Z}]$ are independent and identically distributed as random variables $[X\ Y\ W\ Z]$. Hence,

$$X = (1 - Z)Y + ZW \qquad \text{or} \qquad X = Y + ZR \tag{2.1}$$

where $R = W - Y$ is the auditing error. Let us underline that in order to simplify our considerations, we assume that $R \geq 0$. Let us note that the following definition is permissible: $W = f(Y, \boldsymbol{B})$, where function $f(.)$ is not necessary linear and $\boldsymbol{B} = [B_1 ... B_a]$ are some non necessarily observable variables representing circumstances which generate the inaccurate observation. The idea of such type model construction is due to Marazzi and Tillé (2016). They proposed the model of accounting amount dependent on logit function of auxiliary variables that are used to describe different aspects of accounting transactions.

In practice, all values of $\boldsymbol{X}$ are known before auditing process. Observations $\boldsymbol{x}$ of $\boldsymbol{X}$ are treated as specific auxiliary data. Auditing process leads to observation of values of Z_i, Y_i, and W_i, $i \in U$. Let $\bar{X} = \frac{1}{N}\sum_{i \in U} X_i$, $\bar{Y} = \frac{1}{N}\sum_{i \in U} Y_i$, $\bar{W} = \frac{1}{N}\sum_{i \in U} W_i$. Their values will be denoted by $\bar{x}$, $\bar{y}$, and $\bar{w}$, respectively.

The probability distribution of the matrix $[\boldsymbol{X}\ \boldsymbol{Y}\ \boldsymbol{W}\ \boldsymbol{Z}]$ can be treated as the population model (see, e.g., Cassel et al. (1977) or Ghosh and Meeden (1997)). This assumption in the context of auditing inference is considered, e.g., by Cox and Snell (1979). Below, we show the more specific model of generation accounting data based on the mixture of probability distributions.

Particularly, let us assume that values of variables Y_i and W_i are fixed. It means that $Y_i = y_i$ and $W_i = w_i$, $i \in U$ with probability one. Hence, only $Z_i = z_i$, $i \in U$ are random. In this case, we can show that $X_i = (1 - Z_i)y_i + Z_i w_i$ for $i \in U$ and $E(\bar{X}) = (1 - p)\bar{y} + p\bar{w}$.

When we additionally assume that random variables Z_i, $i \in U$ are fixed, the superpopulation model simplifies to the fixed and finite population. The mean of book values is $\bar{x} = p_0\bar{w} + p_1\bar{y}$, where $p_k = N_k/N$, N_k is size of sets U_k, $k = 0, 1$, $U = U_0 \cup U_1$. U_0 is the subpopulation where true values of accounting amounts are observed while U_1 is the subpopulation with observations of accounting amounts contaminated with errors.

Let $\tau = E(\bar{X} - \bar{Y})$ be the expected mean accounting error. Audit purpose is inference on τ or on the expected total accounting error $N\tau = E(\sum_{i\in U} X_i - \sum_{i\in U} Y_i)$. In particular, when we assume that τ_0 is the admissible mean accounting error then the inference reduces to testing the following hypothesis:

$$H_0: \quad \tau \leq \tau_0, \qquad H_1: \quad \tau = \tau_1 > \tau_0. \tag{2.2}$$

where τ_1 is unadmissible level of the mean accounting error. The probability distribution of X_i is treated as the mixture of distributions of Y_i and W_i, $i \in U$. Let us assume that $F_0(y|\boldsymbol{\theta}_0)$ is the probability distribution function of the random variable Y and $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}_0$ where $\boldsymbol{\Theta}_0$ is the parameter space. The distribution function of W is denoted by $F_1(w|\boldsymbol{\theta}_1)$ where $\boldsymbol{\theta}_1 \in \boldsymbol{\Theta}_1$. The conditional distribution functions of X will be defined by: $F(x|Z = 0) = F_0(x|\boldsymbol{\theta_0})$ and $F(x|Z = 1) = F_1(x|\boldsymbol{\theta_1})$. According to the well-known total probability theorem, we have $F(x) = F(x|Z = 0)P(Z = 0) + F(x|Z = 1)P(Z = 1)$, and finally

$$F(x|\boldsymbol{\theta}) = (1 - p)F_0(x|\boldsymbol{\theta}_0) + pF_1(x|\boldsymbol{\theta}_1), \tag{2.3}$$

where $\boldsymbol{\theta} = \boldsymbol{\theta}_0 \cup \boldsymbol{\theta}_1$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta} = \boldsymbol{\Theta}_0 \cup \boldsymbol{\Theta}_1$ is the parameter space. Hence, the probability distribution of observed accounting amounts is the mixture of the distribution function $F_0(x|\boldsymbol{\theta}_0)$ of the true amounts and the distribution function $F_1(x|\boldsymbol{\theta}_1)$ of the amounts contaminated by errors.

When random variables Y and W are continuous, by differentiating both sides of the equation (2.3), we have

$$f(x|\boldsymbol{\theta}) = (1 - p)f_0(x|\boldsymbol{\theta}_0) + pf_1(x|\boldsymbol{\theta}_1). \tag{2.4}$$

Hence, the probability density of observed accounting amounts is the mixture of the density $f_0(x|\boldsymbol{\theta}_0)$ of the true amounts and the density $f_1(x|\boldsymbol{\theta}_1)$ of the amounts contaminated by errors. In the case of discrete probability distribution, we have

$$P(X = x|\boldsymbol{\theta}) = (1 - p)P_0(Y = x|\boldsymbol{\theta}_0) + pP_1(W = x|\boldsymbol{\theta}_1). \tag{2.5}$$

Let us note that the density function of the accounting error can be written as follows:

$$f_2(r|\boldsymbol{\theta}, p) = pf_3(r|\boldsymbol{\theta})I(r \neq 0) + (1-p)I(0), \tag{2.6}$$

where r is value of R and $I(r) = 1$ when $r \neq 0$ and $I(r) = 0$ when $r = 0$. Models based on the above mixture of distributions was considered by Chen et al. (1998), Kvanli et al. (1998), or Meng (1977).

The basic moments of the random variable X are:

$$E(X) = pE(X|Z=1) + (1-p)E(X|Z=0) = pE(W) + (1-p)E(Y), \tag{2.7}$$

$$\begin{aligned} V(X) = p(1-p)\left((E(X|Z=1) - E(X|Z=0)\right)^2 + \\ + pV(X|Z=1) + (1-p)V(X|Z=0) = \\ = p(1-p)(E(W) - E(Y))^2 + pV(W) + (1-p)V(Y). \end{aligned} \tag{2.8}$$

The random vector $\boldsymbol{X}_s$ is observed in sample s where the objects are controlled. After the auditing process s is split into two disjoint sub-samples s_0 and s_1 where $s_0 \cup s_1 = s$. The set s_1 is of size $n_1 = k$ and the set s_0 is of size $n_0 = n-k$. In sub-sample s_0, there are observed accounting amounts without errors. They are values of the random variables denoted by $\{X_i = Y_i, i \in s_0\}$ or $\boldsymbol{X}_{s_0} = \boldsymbol{Y}_{s_0}$. In the sub-sample s_1, accounting amounts contaminated by errors are observed as values of the random variables $\{W_i, i \in s_1\}$ or $\boldsymbol{X}_{s_1} = \boldsymbol{W}_{s_1}$. Hence, before auditing process, we have observations of the following data:

$$\boldsymbol{X} = (X_i : i \in U) = (\boldsymbol{X}_s, \boldsymbol{X}_{U-s})$$

where

$$\boldsymbol{X}_s = (X_i : i \in s), \quad \boldsymbol{X}_{U-s} = (X_i : i \in U - s)$$

After the auditing process, we have observations of the following data:

$$\boldsymbol{\mathcal{D}} = (\boldsymbol{\mathcal{D}}_s, \boldsymbol{X}_{U-s}), \quad \boldsymbol{\mathcal{D}}_s = ((X_i, Z_i) : i \in s) = (\boldsymbol{Y}_{s_0}, \boldsymbol{W}_{s_1}).$$

Values (outcomes) of $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{D}}_s$, $\boldsymbol{X}$, $\boldsymbol{X}_s$, $\boldsymbol{X}_{U-s}$, $\boldsymbol{Y}_{s_0}$, and $\boldsymbol{W}_{s_1}$ will be denoted by $\boldsymbol{d}$, $\boldsymbol{d}_s$, $\boldsymbol{x}$, $\boldsymbol{x}_s$, $\boldsymbol{x}_{U-s}$, $\boldsymbol{y}_{s_0}$, and $\boldsymbol{w}_{s_1}$ respectively.

In the context of model approach, our purpose is to test the hypothesis H_0, given by (2.2), about the expected value of the following difference of the sum of observed in the population accounting amounts, and the sum of the true values. On the basis of the equation (2.7), we have

$$\tau = E\left(\bar{X} - \bar{Y}\right) = E(X) - E(Y) = p(E(W) - E(Y))$$

or

$$\tau(\boldsymbol{\theta}) = E(X|\boldsymbol{\theta}) - E(Y|\boldsymbol{\theta}_0) = p(E(W|\boldsymbol{\theta}_1) - E(Y|\boldsymbol{\theta}_0)). \qquad (2.9)$$

The mixture of two Poisson distributions is the particular case of the described model, and it was considered by Wywiał (2016). Below, the mixture of gamma probability distributions is taken into account.

3 Mixture of Gamma Distribution

3.1. Basic Properties The well-known gamma probability distribution we denote by $G(\alpha, \beta)$ where parameters $\alpha > 0$ and $\beta > 0$ are called scale and shape parameters. The expected value and the variance of gamma distribution are equal $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$, respectively. The shape of gamma density distribution does not depend on the scale parameter because its skewness and kurtosis coefficients are equal to $\frac{2}{\sqrt{\beta}}$ and $\frac{6}{\beta}$, respectively.

Let $Y \sim G(a, c)$ and $R \sim G(b, c)$ be independent. Under these assumptions the random variable $W = Y + R \sim G(a+b, c)$. According to the results of the previous subsection, the observed value before the auditing process, we define as the value of a mixture of distributions of the variables Y and W. The equation (2.4) leads to:

$$f(x|a, b, c, p) = pf_1(x|a, b, c) + (1-p)f_0(x|a, c), \qquad (3.1)$$

where

$$f_1(x|a, b, c) = \frac{c^{a+b}}{\Gamma(a+b)} x^{a+b-1} e^{-cx}, \quad f_0(x|a, c) = \frac{c^a}{\Gamma(a)} x^{a-1} e^{-cx}, \quad x > 0.$$

Based on expressions (2.7)–(2.8), we obtain

$$E(X) = \frac{a+pb}{c}, \qquad V(X) = \frac{a+pb+p(1-p)b^2}{c^2}. \qquad (3.2)$$

Usually, in auditing practice, we can expect that values of the true accounting errors and values of the accounting error are measured on the same scale. Moreover, their distributions are right-skewed because small book amounts are more frequent than large book amounts. It is convenient to assume that the book values are additive function of true accounting amounts and accounting errors. Hence, we can expect that the above proposed quite simple model describes accounting data well. More about using gamma distribution in modeling accounting error can be found in the papers by Frost and Tamura (1986) or Tamura (1988). In the next subsections, parameters of the distributions mixture will be estimated by means of the method of moments and the maximum likelihood method. These methods provide consistent estimators of parameters.

3.2. Inference Based on Sample Moments

Case 1: $n = 0$**,** $N > 0$**,** $s = \varnothing$ It means that the sample is not a selected in order to audit observed data. In this situation, the solution $\{p_U(x), a_U(x), b_U(x), c_U(x)\}$ of the equation system:

$$E(X^e) = m_e(x), \quad e = 1, 2, 3, 4, \tag{3.3}$$

where $m_e(x) = \frac{1}{N}\sum_{i \in U} x_i^e$, provides the estimators of the parameter p, a, b, and c. In practice, some appropriate numerical methods like the well-known Newton's one can be applied to solve the equation (3.3).

Now let us consider the hypotheses defined by the expression (2.2) about the function of the parameters $\tau(\theta) = \tau(p, a, b, c)$. In our case on the basis of the expressions (2.9) and (3.2), we have

$$\tau = \tau(p, b, c) = E(X|a, b, c, p) - E(Y|a, c) = \frac{pb}{c}. \tag{3.4}$$

Hence, the parameter τ can be estimated by means of the statistic $\hat{\tau}_1 = \frac{p_U(x) b_U(x)}{c_U(x)}$. On the basis of the well-known asymptotic properties of the method of moment (see, e.g. Cramér (1962)), we infer that when the size N is sufficiently large, the statistic has approximately normal distribution $N(\tau, O(N^{-3/2}))$. Let $Q_U(\boldsymbol{\mathcal{D}})$ be a consistent (e.g., bootstrap type) estimator of the variance of $\hat{\tau}_1$. This let us propose the following test statistic of the hypothesis (2.2):

$$\hat{G}_1 = \frac{\hat{\tau}_1 - \tau_0}{\sqrt{Q_U(\boldsymbol{\mathcal{D}})}}. \tag{3.5}$$

When the hypothesis (2.2) is true and N is sufficiently large, the statistic $\hat{G}_1$ has approximately standard normal distribution. So, under the preassigned significance level, it lets us make decision on eventual rejecting the hypothesis (2.2). Moreover, let us underline that the just presented testing problem is based only on observed book accounting amounts. So, the auditing process of those observation is not needed, provided the assumed model defined as the mixture of two gamma distribution is true.

Case 2: $n - k > 0$, $k > 0$ and $N - n > 0$ In this situation, we assume that after the auditing process, we have observations of data $\boldsymbol{\mathcal{D}}$ where $s_0 \neq \varnothing$ and $s_1 \neq \varnothing$. In this case, we have

$$\begin{cases} E(X) = \frac{a}{c} + p\frac{b}{c}, \\ E(Y) = \frac{a}{c}, \\ E(R) = \frac{b}{c}, \\ V(Y) = \frac{a}{c^2}. \end{cases} \tag{3.6}$$

Let

$$\bar{X}_\omega = \frac{1}{\text{Card}(\omega)} \sum_{k \in U-s} X_k, \quad v_\omega(X) = \frac{1}{\text{Card}(\omega) - 1} \sum_{k \in \omega} (x_k - \bar{X}_\omega)^2,$$

where Card(ω) is size of ω and $\omega = U$, $\omega = U - s$, $\omega = s$, $\omega = s_i$, i=0,1. Particularly, $\bar{X}_{s_0} = \bar{Y}_{s_0}$, $\bar{X}_{s_1} = \bar{W}_{s_1}$, $v_{s_1}(X) = v_{s_1}(Y)$, and $v_{s_1}(X) = v_{s_1}(W)$.

After replacing the moments $E(X)$, $V(Y)$, $E(Y)$, E(R), and $V(Y)$ with the sample moments $\bar{X}_{U-s}$, $V_{U-s}(X)$, $\bar{Y}_{s_0}$, $\bar{R}_{s_1} = \bar{W}_{s_1} - \bar{Y}_{s_1}$, and $V_{s_0}(Y)$, respectively, the appropriate algebraic operations let us evaluate the following estimators of parameters p, a, b, and c

$$\begin{cases} P_U = \frac{\bar{X}_{U-s} - \bar{Y}_{s_0}}{\bar{R}_{s_1}}, \\ A_{s_0} = \frac{\bar{Y}_{s_0}^2}{V_{s_0}(Y)}, \\ B_s = \frac{\bar{Y}_{s_0} \bar{R}_{s_1}}{V_{s_0}(Y)}, \\ C_{s_0} = \frac{\bar{Y}_{s_0}}{V_{s_0}(Y)} \end{cases} \tag{3.7}$$

provided denominators of the above fractions are positive. In this case, the unbiased estimator of τ is as follows:

$$\hat{\tau}_2 = \bar{X}_{U-s} - \bar{Y}_{s_0}. \tag{3.8}$$

This let us construct the following test statistic of hypothesis H_0, expressed by (2.2):

$$\hat{G}_2 = \frac{\hat{\tau}_2 - \tau_0}{\sqrt{\frac{V_{U-s}(X)}{N-n} + \frac{V_{s_0}(Y)}{n_0}}}, \tag{3.9}$$

where $n_0 = n - k$. On the basis of the well-known properties, the probability distribution of sample moments (see, e.g., Cramér 1962), we can prove that $\hat{G}_2 \sim N(0;1)$ when H_0 is true and $N \to \infty$, $n_0 \to \infty$.

Case 3: $k = 0$**,** $N \geq n$ In this situation, we assume that only the true values are observed in the sample s. So, it means that the sub-sample s_1 is empty and $n_0 = n$. In this situation, we have $\boldsymbol{\mathcal{D}} = \boldsymbol{Y}_s \cup \boldsymbol{X}_{U-s}$. Hence, we have $N - n$ book amounts and n observations of the true amounts. The estimators are determined on the basis of the following equation system:

$$\begin{cases} E(X) = \frac{a}{c} + p\frac{b}{c} \\ V(X) = \frac{a + pb + p(1-p)b^2}{c^2} \\ E(Y) = \frac{a}{c} = \bar{y}_s \\ V(Y) = \frac{a}{c^2} = v_s(y) \end{cases} \tag{3.10}$$

After replacing the moments $E(X)$, $V(X)$, E(Y), and $V(Y)$ with the sample moments $\bar{X}_{U-s}$, $V_{U-s}(X)$, $\bar{Y}_s$, and $V_s(Y)$, respectively, we derive the following estimators of parameters p, a, b, and c:

$$\begin{cases} \hat{P}_U = \frac{(\bar{X}_{U-s}-\bar{Y}_s)^2}{(\bar{X}_{U-s}-\bar{Y}_s)^2+V_{U-s}(X)-\frac{\bar{X}_{U-s}}{\bar{Y}_s}V_s(Y)} \\ \hat{A}_s = \frac{\bar{Y}_s^2}{V_s(Y)} \\ \hat{B}_{U-s} = \frac{\bar{Y}_s(\bar{X}_{U-s}-\bar{Y}_s)^2+\bar{Y}_s V_{U-s}(X)-\bar{X}_{U-s}V_s(Y)}{V_s(y)(\bar{X}_{U-s}-\bar{Y}_s)} \\ \hat{C}_s = \frac{\bar{Y}_s}{V_s(Y)} \end{cases} \tag{3.11}$$

provided denominators of the above fractions are positive. In this case, the unbiased estimator of τ is as follows:

$$\hat{\tau}_3 = \bar{X}_{U-s} - \bar{Y}_s. \tag{3.12}$$

The test statistic of hypothesis H_0, expressed by (2.2), is as follows:

$$\hat{G}_3 = \frac{\hat{\tau}_3 - \tau_0}{\sqrt{\frac{V_{U-s}(x)}{N-n} + \frac{V_s(y)}{n}}} \tag{3.13}$$

We can show that $\hat{G}_3 \sim N(0;1)$ when H_0 is true and $N \to \infty$, $n \to \infty$.

Case 4: $k = n$**,** $N \geq n$ Now, the true values are not observed in the sample s. So, it means that the sub-sample s_0 is empty and $n_0 = 0$. In this situation, $\boldsymbol{\mathcal{D}} = \boldsymbol{W}_s \cup \boldsymbol{X}_{U-s}$. Hence, we have $N - n$ book amounts and n observations of the amounts contaminated by errors. In this situation, the estimators are evaluated based on the following equation system:

$$\begin{cases} E(X) = \frac{a}{c} + p\frac{b}{c}, \\ V(X) = \frac{a+pb+p(1-p)b^2}{c^2}, \\ E(W) = \frac{a+b}{c}, \\ V(W) = \frac{a+b}{c^2}. \end{cases} \tag{3.14}$$

After replacing the moments $E(X)$, $V(X)$, E(W), and $V(W)$ with the sample moments $\bar{X}_{U-s}$, $V_{U-s}(X)$, $\bar{W}_s$, and $V_s(W)$, respectively, the appropriate algebraic operations let us derive the following estimators of p, a, b, and c:

$$\begin{cases} \tilde{P}_U = \dfrac{\left(\frac{V_{U-s}(X)}{V_s(W)}\bar{W}_s - \bar{X}_{U-s}\right)^2}{\left(\frac{V_{U-s}(X)}{V_s(W)}\bar{W}_s - \bar{X}_U\right)^2 + \left(\bar{W}_s - \bar{X}_{U-s}\right)^2}, \\ \tilde{A}_U = \frac{\bar{W}_s}{V_s(W)}\left(\dfrac{\left(\frac{V_{U-s}(X)}{V_s(W)}\bar{W}_s - \bar{X}_{U-s}\right)^2 + \left(\bar{W}_s - \bar{X}_{U-s}\right)^2}{\bar{W}_s - \bar{X}_{U-s}} - \bar{W}_s\right), \\ \tilde{B}_U = \frac{\bar{W}_s}{V_s(W)} \dfrac{\left(\frac{V_{U-s}(X)}{V_s(W)}\bar{W}_s - \bar{X}_{U-s}\right)^2 + \left(\bar{W}_s - \bar{X}_{U-s}\right)^2}{\bar{W}_s - \bar{X}_{U-s}}, \\ \tilde{C}_s = \frac{\bar{W}_s}{V_s(W)}. \end{cases} \tag{3.15}$$

provided denominators of the above fractions are positive. In this case, estimator of τ is:

$$\hat{\tau}_4 = \frac{\left(\frac{V_{U-s}(X)}{V_s(W)}\bar{W}_s - \bar{X}_{U-s}\right)^2}{\bar{W}_s - \bar{X}_{U-s}} = \bar{W}_s - \bar{X}_{U-s} - 2\bar{W}_s\left(1 - \frac{V_{U-s}(X)}{V_s(W)}\right) + \frac{\bar{W}_s^2}{\bar{W}_s - \bar{X}_{U-s}}\left(1 - \frac{V_{U-s}(X)}{V_s(W)}\right)^2 \tag{3.16}$$

provided $\bar{W}_s > \bar{X}_{U-s}$. Let Q be a consistent estimator of the variance of $\hat{\tau}_4$. The statistic $\hat{G}_4 = \frac{\hat{\tau}_4 - \tau_0}{\sqrt{Q}} \sim N(0,1)$ provided H_0 is true and $N \to \infty$, $n \to \infty$.

3.3. Testing on the Basis of the Likelihood Function Expression (2.4) and the above results lead to the following likelihood function:

$$L\left(\boldsymbol{d}|\boldsymbol{\theta}\right) = L\left(\boldsymbol{d}_s|\boldsymbol{\theta}\right) L\left(\boldsymbol{x}_{U-s}|\boldsymbol{\theta}\right) \tag{3.17}$$

where

$$L\left(\boldsymbol{d}_s|\boldsymbol{\theta}\right) = \prod_{i\in s} p^{z_i} f_1^{z_i}(x_i|\boldsymbol{\theta}_1)(1-p)^{1-z_i} f_0^{1-z_i}(x_i|\boldsymbol{\theta}_0),$$

$$L\left(x_{U-s}|\boldsymbol{\theta}\right) = \prod_{i\in U-s} f(x_i|\boldsymbol{\theta}).$$

If $z_i = 0$ ($z_i = 1$), then $x_i = y_i$ ($x_i = w_i$). The logarithm of the likelihood function is as follows:

$$l\left(\boldsymbol{d}|\boldsymbol{\theta}\right)) = kln(p) + (n-k)ln(1-p) + \sum_{i\in s_1} ln(f_1(x_i|\boldsymbol{\theta}_1)) + \sum_{i\in s_0} ln(f_0(x_i|\boldsymbol{\theta}_0)) + \sum_{i\in U-s} ln(f(x_i|\boldsymbol{\theta})). \quad (3.18)$$

The expressed by equation (2.2) hypotheses can be verified by means of the well-known likelihood ratio test on the basis of the following statistic:

$$\lambda = \frac{\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta},\tau(\boldsymbol{\theta})=\tau_0} L\left(\boldsymbol{\mathcal{D}}|\theta\right)}{\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L\left(\boldsymbol{\mathcal{D}}|\boldsymbol{\theta}\right)}. \quad (3.19)$$

We can expect that when hypothesis H_0 is true and N, n, $N-n$, n_0, and $n-n_0$ are sufficiently large then statistic $t = -2ln(\lambda)$ is well approximated by the chi-square distribution with one degree of freedom(see, e.g., Silvey (1959)). Hypothesis H_0 is rejected if t is significantly large. Moreover, let us note that the ratio likelihood function let us test hypothesis H_0' : $\tau = \tau_0$ and $p = p_0$. In this case, the nominator of the above ratio is as follows: $\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta},\tau(\theta)=\tau_0,p=p_0} L\left(\boldsymbol{\mathcal{D}}|\boldsymbol{\theta}\right)$ and the log-likelihood ratio statistic has asymptotically chi-square distribution with two degree of freedom provided H_0' is true.

Inference on accounting error parameters can be based on the distribution mixture defined by expression (2.6), when accounting errors $\boldsymbol{R} = \{R_i = X_i - Y_i, i \in s\}$ are only functions of $\boldsymbol{\mathcal{D}}_s$ observed after auditing process. In this case, the log-likelihood function is:

$$l_{\#}\left(\boldsymbol{R}|\boldsymbol{\theta}\right) = kln(p) + (n-k)ln(1-p) + \sum_{i\in s_1} f_2(r_i|\boldsymbol{\theta}).$$

Hence, in this case, the data about the book variable observed as values of vector $\boldsymbol{X}_{U-s}$ are not taken into account. Inference based on the likelihood function $l_{\#}$ is similar to the previous case.

In the considered case of gamma type distribution of the data, the defined by expression (3.18) log-likelihood function is as follows:

$$l(\boldsymbol{d}_U, a, b, c, p) = k\ ln(p) + (n-k)ln(1-p) + Na\ ln(c) + kb\ ln(c) + \\ - k\ ln(\Gamma(a+b)) - (n-k)ln(\Gamma(a)) + (a-1)\sum_{j\in U} ln(x_j) + b\sum_{j\in s_1} ln(x_j) + \\ - c\sum_{j\in U} x_j + \sum_{j\in U-s} ln\left(\varphi(a,b,c,p,x_j)\right), \quad (3.20)$$

where

$$\varphi(a,b,c,p,x_j) = \frac{1-p}{\Gamma(a)} + \frac{p(cx_j)^b}{\Gamma(a+b)}.$$

The derivatives of that function are as follows:

$$\frac{\partial l}{\partial a} = Nln(c) - \psi(a+b) - (n-k)\psi(a) + \sum_{j\in U} ln(x_j) +$$
$$- \sum_{j\in U-s} \varphi^{-1}(a,b,c,p,x_j)\left(\frac{(1-p)\psi(a)}{\Gamma(a)} + \frac{p(cx_j)^b\psi(a+b)}{\Gamma(a+b)}\right)$$

where

$$\psi(u) = -\gamma - \frac{1}{u} + \sum_{t=1}^{\infty}\left(\frac{1}{t} - \frac{1}{u+t}\right) = \frac{1}{\Gamma(u)}\frac{\partial\Gamma(u)}{\partial u},$$

$\gamma = 0.57722$ is Euler - Mascheroni constant,

$$\frac{\partial l}{\partial b} = kln(c) - k\psi(a+b) + \sum_{j\in s_1} ln(x_j)$$
$$+ \frac{pc^b}{\Gamma(a+b)} \sum_{j\in U-s} \varphi^{-1}(a,b,c,p,x_j)\left(ln(cx_j) - \psi(a,b,p)\right) x_j^b,$$

$$\frac{\partial l}{\partial c} = \frac{Na+kb}{c} - \sum_{j\in U} x_j + \frac{pbc^{b-1}}{\Gamma(a+b)} \sum_{j\in U-s} \varphi^{-1}(a,b,c,p,x_j)x_j^b,$$

$$\frac{\partial l}{\partial p} = \frac{k}{p} - \frac{n-k}{1-p} + \sum_{j\in U-s} \varphi^{-1}(a,b,c,p,x_j)\left(\frac{(cx_j)^b}{\Gamma(a+b)} - \frac{1}{\Gamma(a)}\right).$$

Particularly, if $s = U$ or equivalently $n = N$ and $1 < k < n$:

$$\begin{cases} \frac{\partial l}{\partial a} = nln(c) + \sum_{j\in s} ln(x_j) - k\psi(a+b) - (n-k)\psi(a), \\ \frac{\partial l}{\partial b} = kln(c) + \sum_{j\in s_1} ln(x_j) - k\psi(a+b), \\ \frac{\partial l}{\partial c} = \frac{na}{c} + \frac{kb}{c} - \sum_{j\in s} x_j, \\ \frac{\partial l}{\partial p} = \frac{k}{p} - \frac{n-k}{1-p}. \end{cases}$$

On the basis of the above results, the maximum likelihood estimators of the parameters a, b, p and either c or τ can be derived but only by means of the appropriate numerical methods like gradient type ones. This lets us construct the test statistic in order to verify hypothesis (2.2). More details about inference on mixtures of probability distributions can be found in the book by McLachlan and Peel (2000), where, e.g., the well-known EM algorithm is used to evaluate the maximum likelihood estimators.

3.4. Monte Carlo Tests In the last subsection, testing hypotheses is practically possible when sample sizes are large. For moderate and small sample sizes, hypotheses can be tested using the Monte Carlo method (see, e.g., Davison and Hinkley (1997), Dimitrov et al. (2003), Dufour (2006), Dufour and Khalaf (2001), Hall (1992), and MacKinnon (2007). In order to apply this method, all of the considered models need to be reparametrized. This lets us simplify the inference on total error amount. Using Eq. (3.4), we replace parameter c with the right side of the equation: $c = \frac{pb}{\tau}$. This transforms the model given by (3.1) to the following:

$$f(x|a,b,\tau,p) = pf_1(x|a,b,\tau) + (1-p)f_0(x|a,\tau)$$

where:

$$f_0(x|a,\tau) = \frac{(pb)^a}{\tau^a\Gamma(a)}x^{a-1}e^{-p\frac{b}{\tau}x}, \quad f_1(x|a,\tau) = \frac{(pb)^{a+b}}{\tau^{a+b}\Gamma(a+b)}x^{a+b-1}e^{-p\frac{b}{\tau}x}.$$

Firstly, let us focus on *Case 2* considered in Section 3.2. Our purpose is to test hypotheses defined by expression (2.2). Parameter a, b, p, and τ are estimated using statistics defined by expressions (3.7) and (3.8). Dufour (2006) showed that Monte Carlo tests obtained after replacing unknown parameters (in our case a, b, p) with consistent estimators are valid. Therefore, data $\boldsymbol{d}^{(0,i)} = \left(\boldsymbol{y}_{s_0}^{(0,i)}, \boldsymbol{w}_{s_1}^{(0,i)}, \boldsymbol{x}_{U-s}^{(0,i)}, \right)$, $i = 1, ..., m$, are generated according to densities $f_0(y|a_{s_0}, \tau_0)$, $f_1(w|a_{s_0}, b_s, \tau_0)$ and $f(x|a_{s_0}, b_s, \tau_0, p_U)$. Data $\boldsymbol{d}^{(1,i)} = \left(\boldsymbol{y}_{s_0}^{(1,i)}, \boldsymbol{w}_{s_1}^{(1,i)}, \boldsymbol{x}_{U-s}^{(1,i)}, \right)$, $i = 1, ..., m$ are generated based on $f_0(y|a_{s_0}, \tau_1)$, $f_1(w|a_{s_0}, b_s, \tau_1)$ and $f(x|a_{s_0}, b_s, \tau_1, p_U)$. Next, we evaluate the following:

$$\hat{g}_2^{(e,i)} = \frac{\tau^{(e,i)} - \tau_0}{\sqrt{\frac{v_{U-s}\left(\boldsymbol{x}^{(e,i)}\right)}{N-n} + \frac{v_{s_0}\left(\boldsymbol{y}^{(e,i)}\right)}{n_0}}}, \qquad \tau^{(e,i)} = \bar{X}_{U-s}^{(e,i)} - \bar{Y}_{s_0}^{(e,i)}, \quad e = 0, 1.$$

The sequence $\{\hat{g}_2^{(e,i)}, i = 1, ..., m\}$ approximates the distribution of the test statistic $\hat{G}_2$, given by expression (3.9), under the assumption that hypothesis H_e is true, $e = 0, 1$. The p value of the test denoted by Dufour and Khalaf (2001) by $\hat{\alpha}$ and defined as $\hat{\alpha} = \eta_e$ for $e = 0$ where:

$$\eta_e = \frac{m\omega_e}{m+1}, \quad \omega_e = \frac{1}{m}\sum_{i=1}^{m} I(\hat{g}_2^{(e,i)}), \quad I(\hat{g}_2^{(e,i)}) = \left\{ \begin{array}{l} 1,\ if\ g \geq \hat{g}_2 \\ 0,\ if\ g < \hat{g}_2 \end{array} \right., \ e = 0, 1.$$

ω_e is equal to the frequency of appearing inequalities $\hat{g}_2^{(e,i)} \geq \hat{g}_2$, $i = 1, ..., m$, $e = 0, 1$. The power of the test is assessed by $\hat{\beta} = \eta_1$. Moreover, $\hat{\alpha} = \omega_0$, and $\hat{\beta} = \omega_1$ for large m. When $\hat{\alpha} \leq \alpha$, hypothesis H_0 is rejected, but this decision can be wrong with the risk of incorrect rejection equal to probability α. If $\hat{\alpha} > \alpha$, hypothesis H_0 is accepted, and this decision can be wrong with the risk of incorrect acceptance equal to $(1 - \hat{\beta})$.

The equivalent inference is as follows. When α is the significance level of the test, then the quantile of order $(1 - \alpha)$ of the test statistic distribution evaluated based on the ordered sequence $\{\hat{g}_2^{(0,i)}, i = 1, ..., m\}$ approximates the critical value of the test. This is denoted by $g_{2,\alpha}$. Hence, when $\hat{g}_2 \geq g_{2,\alpha}$, then hypothesis H_0 is rejected but this decision is wrong with probability equal to α. When $\hat{g}_2 < g_{2,\alpha}$, hypothesis H_0 is accepted but this decision is wrong with probability equal to $(1 - \hat{\beta})$.

The presented procedure can be applied to testing the cases 1, 3, and 4 considered in Section 3.2.

After substituting c for $\frac{pb}{\tau}$ in expression (3.20), we have:

$$
\begin{aligned}
l(\boldsymbol{d}, a, b, \tau, p) &= k\, ln(p) + (n-k) ln(1-p) + Na\, ln(c) + kb\, ln(c) + \\
&- k\, ln(\Gamma(a+b)) - (n-k) ln(\Gamma(a)) + (a-1)\sum_{j\in U} ln(x_j) + b\sum_{j\in s_1} ln(x_j) + \\
&- \frac{pb}{\tau}\sum_{j\in U} x_j + \sum_{j\in U-s} ln\left(\varphi(a, b, \tau, p, x_j)\right) \qquad (3.21)
\end{aligned}
$$

where

$$
\varphi(a, b, \tau, p, x_j) = \frac{1-p}{\Gamma(a)} + \frac{p^{b+1}(bx_j)^b}{\tau^b \Gamma(a+b)}.
$$

This lets us evaluate the new estimator of τ based on likelihood function $l(\boldsymbol{d}, a, b, \tau, p)$. This reparametrization of the log-likelihood function leads to a simplification the test of the above formulated hypotheses. The logarithm of the likelihood ratio test statistic takes the following form:

$$
t = 2\left(l(\boldsymbol{d}, \hat{a}, \hat{b}, \hat{\tau}, \hat{p}) - l(\boldsymbol{d}, \tilde{a}, \tilde{b}, \tau_e, \tilde{p})\right)
$$

where $(\hat{a}, \hat{b}, \hat{\tau}, \hat{p})$ maximizes the function expressed by (3.21) and $(\tilde{a}, \tilde{b}, \tilde{p})$ maximizes function $l(\boldsymbol{d}, a, b, \tau_0, p)$. According to density $f(x|\tilde{a}, \tilde{b}, \tau_e, \tilde{p})$, data $\boldsymbol{d}^{(e,i)}$, $e = 0, 1$ is generated. This and expression (3.21) lets us evaluate the test statistic values:

$$
t_i^{(e)} = 2\left(l(\boldsymbol{d}^{(e,i)}, \hat{a}^{(i)}, \hat{b}^{(i)}, \hat{\tau}^{(i)}, \hat{p}) - l(\boldsymbol{d}^{(e,i)}, \tilde{a}^{(i)}, \tilde{b}^{(i)}, \tau_e, \tilde{p}^{(i)})\right), \quad e = 0, 1
$$

where $(\hat{a}^{(i)}, \hat{b}^{(i)}, \hat{\tau}^{(i)}, \hat{p}^{(i)})$ maximizes the function expressed by (3.21), and $(\tilde{a}^{(i)}, \tilde{b}^{(i)}, \tilde{p}^{(i)})$ maximizes function $l(\boldsymbol{d}^{(e,i)}, a, b, \tau_e, p)$. The critical value, denoted by t_α, of the log-likelihood test under the significance level α is defined as the quantile of order $(1-\alpha)$ of observations $\{t_i^{(0)}, i = 1, ..., m\}$. The approximate power of the test, denoted by $\hat{\beta}$, is equal to the frequency of appearing inequalities $t_i^{(1)} \geq t_\alpha$, $i = 1, ..., m$. Finally, when $t \geq t_\alpha$ hypothesis H_0 is rejected, but this decision is wrong with probability α. If $t < t_\alpha$ hypothesis H_0 is accepted, and this decision is wrong with probability $(1-\hat{\beta})$.

The p value of the test is approximated by means of the frequency of appearing inequalities $t_i^{(0)} \geq t$, $i = 1, ..., m$. The power of the test is assessed by the frequency of appearing inequalities: $t_i^{(1)} \geq t$, $i = 1, ..., m$. If $\hat{\alpha} \leq \alpha$, then hypothesis H_0 is rejected. This decision is wrong with probability α. When $\hat{\alpha} > \alpha$, then hypothesis H_0 is accepted. This decision is wrong with probability $(1-\hat{\beta})$.

The above procedure can be treated as a generalization of the Monte Carlo test based on the likelihood ratio test considered by Dimitrov et al. (2003). According to their idea, the next test statistic is as follows:

$$t_* = 2\left(l(\boldsymbol{d}, \tilde{a}_0, \tilde{b}_0, \tau_0, \tilde{p}_)) - l(\boldsymbol{d}, \tilde{a}_1, \tilde{b}_1, \tau_1, \tilde{p}_1)\right)$$

where $(\tilde{a}_e, \tilde{b}_e, \tilde{p}_e)$ maximizes function $l(\boldsymbol{d}, a, b, \tau_e, p)$, $e = 0, 1$. According to density $f(x|\tilde{a}_e, \tilde{b}_e, \tau_e, \tilde{p}_e)$, we generate data $\boldsymbol{d}^{(e,i)}$, $e = 0, 1$. This and expression (3.21) lets us evaluate the test statistic values:

$$t_{*,i}^{(e)} = 2\left(l(\boldsymbol{d}^{(e,i)}, \tilde{a}_0^{(i)}, \tilde{b}_0^{(i)}, \tau_0, \tilde{p}_0^{(i)}) - l(\boldsymbol{d}^{(e,i)}, \tilde{a}_1^{(i)}, \tilde{b}_1^{(i)}, \tau_1, \tilde{p}_1^{(i)})\right), \quad e = 0, 1$$

where $i = 1, ..., m$. $(\tilde{a}_0^{(i)}, \tilde{b}_0^{(i)}, \tilde{p}_0^{(i)})$ and $(\tilde{a}_1^{(i)}, \tilde{b}_1^{(i)}, \tilde{p}_1^{(i)})$ maximizes functions $l(\boldsymbol{d}^{(e,i)}, a, b, \tau_0, p)$ and $l(\boldsymbol{d}^{(e,i)}, a, b, \tau_1, p)$, respectively. Now, the critical value, denoted by $t_{*\alpha}$ under the significance level α, is equal to the quantile of order $(1-\alpha)$ of observations $\{t_{*,i}^{(0)}, i = 1, ..., m\}$. The approximate power of the test, denoted by $\tilde{\beta}$, is equal to the frequency of appearing inequalities $t_{*,i}^{(1)} \geq t_{*\alpha}$, $i = 1, ..., m$. Finally, when $t_* \geq t_{*\alpha}$, hypothesis H_0 is rejected, but this decision is wrong with probability α. If $t_* < t_{*\alpha}$, hypothesis H_0 is accepted, and this decision is wrong with probability $(1-\tilde{\beta})$. The inference based on the p value is similar to the previously considered tests.

Example 1. *The population of $N = 4000$ invoices from an anonymous firm was considered. A simple sample of $n = 200$ invoices was selected in order to estimate the parameters of the model. After auditing, $n_1 = 62$ invoices had to be corrected. Therefore, $n_0 = 138$. According to expression (3.7) we evaluated: $a_{s_0} = 1.8802$, $b_s = 1.2402$, $p_U = 0.1570$, $\hat{\tau}_2 = 820.1042$. The hypotheses are: $H_0 : \tau = \tau_0 = 800$ against $H_1 : \tau = \tau_1 = 960$. The value* *of the test statistic, defined by (3.9), is:* $\hat{g}_2 = 0.047$*. Next, according to the simulation procedure, the data were replicated 1000 times. This, under the assessed significance levels* α*: 0.05, 0.1 and 0.2, lets us evaluate the critical values equal to 1.7893, 1.1312 and 0.8834, respectively. Hence, under these significance levels, hypothesis* H_0 *is accepted. For the considered sequence of significance levels the approximated powers of the test are: 0.087, 0.172 and 0.305. Therefore, for instance for* $\alpha = 0.2$*, the acceptance of* H_0 *could be the wrong decision with probability 0.695. Hence, the size of the sample should be increased.*

4 Conclusions

In this paper, we show how to estimate parameters of interest by means of the method of moments or the likelihood method. Particularly, the results let us construct confidence intervals or the statistics to testing hypotheses on the mean or the total amount error with assumed risk of incorrect rejection H_0 (significant level). Moreover, it is possible to control the risk of incorrect acceptance of H_0. Let us underline that on the basis of analysis of case 1 in Section 3.2, we can say that it is possible to make inference on error parameters even when the sample is not selected. Of course, it is possible provided the assumed mixture distribution model is true. Under the same assumption the analysis of the cases 3 and 4 in Section 3.2 lets us conclude that we can even make an inference on the total error amount when in the sample only the true accounting amounts are observed or only the amounts contaminated by errors are observed.

Gamma mixture model was considered in details. The model assumes that scale parameter of the mixture components are the same. Of course, we can consider more general case without that assumption when, e.g., scale of accounting errors is significantly smaller than the scale of the true accounting amounts. Particularly, when $a = b = 1$, the mixture of two simple exponential distribution can be considered (see, e.g., McLachlan and Peel (2000)).

The considered maximum likelihood estimators are usually the solutions of the systems of the non-linear equations. In order to calculate those solutions, some numerical methods have to be used. In general, getting the solutions of such equation systems is possible thanks of the computer programs like "R" or "Mathematica." These problems will be considered in separate papers where the powers of the tests on total error amount will also be analyzed.

In general, except the gamma distribution the following mixtures of probability distributions seem to be useful: mixtures of Pareto distributions, lognormal, or mixtures of Pearson's type distributions with positive skewness. From other point of views, the types of distributions of true accounting amounts and accounting amounts contaminated with errors do not have be the same. For instance, we can consider model of tax data as mixture of the gamma distribution and the distribution defined as the gamma one multiplied by, e.g., the beta distribution on interval $(0;1)$.

Application of model-design approach to analysed problem lets us use data observed in samples drawn from population U according to, e.g., well-known dollar sampling design. This topic in the context of mixture of two Poisson distribution functions is consider by Wywiał (2016). Moreover, this approach let us consider the random vectors $[Y_i\ W_i\ Z_i]$ for $i \in U$ which are dependent or have not the same probability distribution.

Funding Information. The project is supported by the grant of the National Science Centre, Poland, DEC-2012/07/B/HS4/03073.

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Paper received: 1 July 2015.