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# On the Structure of UMVUEs

Abram M. Kagan
*University of Maryland, College Park, USA*
Yaakov Malinovsky
*University of Maryland, Baltimore County, Baltimore, USA*

**Abstract**

In all setups when the structure of UMVUEs is known, there exists a subalgebra $\mathcal{U}$ (MVE-algebra) of the basic $\sigma$-algebra such that all $\mathcal{U}$-measurable statistics with finite second moments are UMVUEs. It is shown that MVE-algebras are, in a sense, similar to the subalgebras generated by complete sufficient statistics. Examples are given when these subalgebras differ, in these cases a new statistical structure arises.

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## 1 Introduction

Among C. R. Rao's major contributions to the foundations of statistical inference (Cramér-Rao inequality, Rao-Blackwellization, Rao's score test, to name the best known), the following observation in Rao (1952) is not widely known.

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be a standard statistical model with $(\mathcal{X}, \mathcal{A})$ a measurable space, and let $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ be a family of probability distributions of a random element $X \in (\mathcal{X}, \mathcal{A})$ parameterized by a general parameter $\theta$.

A statistic $\widehat{g}(X)$ with $E_\theta|\widehat{g}(X)|^2 < \infty$, $\theta \in \Theta$ is called a UMVUE (uniformly minimum variance unbiased estimator), more precisely, the UMVUE of $g(\theta) = E_\theta \widehat{g}(X)$ if

$$Var_\theta(\widehat{g}(X)) \leq Var_\theta(\widetilde{g}(X)),\ \forall\, \theta \in \Theta$$

for any statistic $\widetilde{g}(X)$ (with finite second moment) with $E_\theta \widetilde{g}(X) = E_\theta \widehat{g}(X)$, $\theta \in \Theta$.

If $\widehat{g}_1(X)$ and $\widehat{g}_2(X)$ are UMVUE's, so is their linear combination $c_1\widehat{g}_1(X) + c_2\widehat{g}_2(X)$, $\ c_1, c_2 \in \mathbb{R}$. Rao observed that their product $\widehat{g}(X) = \widehat{g}_1(X)\widehat{g}_2(X)$ is also a UMVUE provided that $E_\theta|\widehat{g}(X)|^2 < \infty,\ \ \theta \in \Theta$ (this is not guaranteed by $E_\theta|\widehat{g}_1(X)|^2 + E_\theta|\widehat{g}_2(X)|^2 < \infty,\ \ \theta \in \Theta$). In all known (to the authors) setups, the class of UMVUE's has the following structure: there exists a subalgebra (an MVE-algebra) $\widehat{\mathcal{A}} \subseteq \mathcal{A}$ such that a statistic $\widehat{g}(X)$ is a UMVUE if and only if $\widehat{g}(X)$ is $\widehat{\mathcal{A}}$-measurable. Combining all this leads to a hypothesis that such is the structure of UMVUEs in general. The results in the paper are closely related to this hypothesis, which, if true, introduces a new statistical structure.

In Section 2 a few setups with known structure of UMVUEs are presented. The first one (see Section 2.1) is classical, in a sense, of a family $\mathcal{P}$ possessing a complete sufficient statistic/subalgebra. The MVE-algebra is complete sufficient subalgebra and every estimable parametric function possesses a UMVUE. The other setups, when the minimal sufficient subalgebra is incomplete, are less known. Plainly, due to the Rao-Blackwell theorem, MVE-algebras are subalgebras of the minimal sufficient subalgebras. In Section 2.2 the setup of a partial complete sufficient subalgebra is considered, that formally generalizes the setup of Section 2.1 (for a relation between partial completeness and completeness see a recent paper (Kagan et al., 2014)). In Section 2.3 a geometric construction leads to a description of the MVE-algebras in case of categorical $X$. Lehmann's example presented in Section 2.4 deals with discrete $X$ taking countable many values and thus is not covered by the results of Section 2.3.

In Section 3 some results for arbitrary (not necessarily generated by complete sufficient statistics) MVE-algebras are proved, showing that the MVE-algebras preserve properties of complete sufficiency.

The existence of UMVUEs in the Nile problem by R. Fisher (see Fisher 1936, 1973) and related problems attracted recently some attention (see Nayak and Sinha (2012)). For its partial solution see Kagan and Malinovsky (2013). For the relation between sufficient statistic, and sufficient subalgebras we refer to Bahadur (1957).

## 2 Setups with Known Structure of UMVUEs

A statistic $\chi(X)$ is called an unbiased estimator of zero (or zero mean statistic) if

$$E_\theta \chi(X) = 0,\ \ \forall\, \theta \in \Theta.$$

A well known result (see, e.g., Lehmann and Casella (1998), p. 85) gives a necessary and sufficient conditions for a statistic $\widehat{g}(X)$ to be a UMVUE. A statistic $\widehat{g}(X)$ is a UMVUE if and only if it is uncorrelated with any zero mean statistic $\chi(X)$ with $E_\theta|\chi(X)|^2 < \infty$, $\forall\, \theta \in \Theta$. Its immediate corollary is that the class of UMVUEs is a linear space: if $\widehat{g}_1$ and $\widehat{g}_2$ are UMVUEs, so is $\widehat{g}(X) = c_1\widehat{g}_1(X) + c_2\widehat{g}_2(X)$ for any constants $c_1$, $c_2$.

In the following subsections, setups with known structure of the class of UMVUEs are presented.

*2.1. Families with Complete Sufficient Subalgebras/Statistics.* Let a statistic $S = S(X)$, $S : (\mathcal{X}, \mathcal{A}) \to (\mathcal{S}, \mathcal{B})$ be a complete sufficient statistic for $\mathcal{P}$, i.e., $E_\theta \chi(S(X)) = 0$, $\forall\, \theta \in \Theta$ implies

$$P_\theta(\chi = 0) = 1 \quad \text{for all} \quad \theta \in \Theta. \tag{1}$$

Equivalently, the subalgebra $\widetilde{\mathcal{A}} = S^{-1}(\mathcal{B})$ of $\mathcal{A}$ is complete sufficient if any $\widetilde{\mathcal{A}}$-measurable $\chi$ with $E_\theta \chi \equiv 0$ implies (1). The concept of completeness is due to Lehmann and Scheffé and its role in the estimation theory is explained by the following. If $\widetilde{\mathcal{A}}$ (resp., $S(X)$) is a complete sufficient subalgebra (resp., statistic) for $\mathcal{P}$, then $\widetilde{\mathcal{A}}$-measurable statistics (resp., statistics depending on $X$ only through $S(X)$) with finite second moment and only they are UMVUEs.

Notice that the factorization theorem does not distinguish between complete and incomplete sufficiency. Proving (or disproving) completeness of a sufficient statistic requires an additional analysis. Lehmann and Scheffé (1950) (see also Lehmann and Casella (1998)) showed that for the natural exponential families of full rank, the minimal sufficient statistic is complete.

*2.2. A Case of Partial Completeness.* The following interesting observation is due to Bondesson (1983). Let $\mathcal{P} = \{P_{\theta,\eta}\,,\, (\theta, \eta) \in \Theta \times \Xi\}$ be a family of distributions on $(\mathcal{X}, \mathcal{A})$ parameterized by a bivariate parameter. Suppose that for any fixed $\eta^* \in \Xi$, a subalgebra $\widetilde{\mathcal{A}}$ (resp., a statistic $S(X)$) is complete sufficient for $\theta$ (i.e., for the family $\mathcal{P}^* = \{P_{\theta,\eta^*},\, \theta \in \Theta\}$). Then any $\widetilde{\mathcal{A}}$-measurable statistic (resp., depending on $X$ through $S(X)$) with finite variance is a UMVUE.

The following models are covered by the Bondesson's result. Suppose that the probability density function of $X = (X_1, X_2)$ is factorized as

$$f\,(x_1, x_2;\, \theta, \eta) = R_1(x_1; \theta) R_2(x_2; \eta) r(x_1; x_2).$$

For any fixed $\eta = \eta^*$, $X_1$ is sufficient for $\theta$. If $X_1$ is complete sufficient, then any statistic $\widehat{g}(X_1)$ is a UMVUE. One can notice that, in general, the parametric function $E_{\theta,\eta}\widehat{g}(X_1) = g(\theta, \eta)$ depends on the whole parameter $(\theta, \eta)$ and not only on the first component. The original example due to

Bondesson (1983) is a sample $X = (X_1, \ldots, X_n)$ from a gamma population with density function

$$f(x;\, \theta, \eta) = \frac{c(\alpha, \beta)}{\theta} (x - \eta)^{\alpha - 1} e^{-\frac{x-\eta}{\beta\theta}},\ x > \eta$$

with $(\theta, \eta) \in \mathbb{R}_+ \times \mathbb{R}$ as parameters, and $\alpha, \beta$ ($\alpha > 0, \alpha \neq 1, \beta > 0$) known. For any fixed $\eta = \eta^*$, $\overline{X}$ is complete sufficient for $\theta$. Any statistic $\widehat{g}(\overline{X})$ (with finite second moment) is the UMVUE for $E_{\theta,\eta}\, \widehat{g}(\overline{X}) = g(\theta, \eta)$. In particular, $\overline{X}$ is the UMVUE for $E_{\theta,\eta}(\overline{X}) = \eta + \alpha\beta\theta$.

Actually, Bondesson (1983) arguments prove a more general result. Namely, if $\widehat{A}$ is an MVE-algebra for $\mathcal{P}^*$ (not necessarily a complete sufficient subalgebra) for any fixed $\eta^*$, then $\widehat{A}$ is an MVE-algebra for $\mathcal{P}$. Indeed, if $\chi(X)$ is a zero-mean statistic with finite second moment for $\mathcal{P}$, it is zero-mean statistic for $\mathcal{P}^*$. So that due to the assumption any $\widetilde{\mathcal{A}}$-measurable $\widehat{g}(X)$ with $E_{\theta,\,\eta^*}|\widehat{g}(X)|^2 < \infty$ one has

$$E_{\theta,\,\eta^*}\,(\widehat{g}\chi) = 0, \quad \forall\, \theta \in \Theta. \tag{2}$$

Since (2) holds for any $\eta^* \in \Xi$, $\widehat{g}$ is a UMVUE for $\mathcal{P}$. One can notice that the reverse is not true, in general. If $\widehat{g}$ is a UMVUE for $\mathcal{P}$, it is not necessarily a UMVUE for $\mathcal{P}^*$. The thing is that the relation $E_{\theta,\,\eta^*}\chi(X) = 0$ for some $\eta^* \in \Xi$ and all $\theta \in \Theta$ does not imply $E_{\theta,\,\eta}\chi(X) = 0$ for all $\eta \in \Xi$, $\theta \in \Theta$, so there are more zero mean statistics for $\mathcal{P}^*$ than for $\mathcal{P}$, in general.

*2.3. UMVUEs from Categorical Data.* Let $X$ be a categorical random variable whose values may be taken as $1, 2, \ldots, N$. The distribution of $X$ is given by

$$P(X = k; \theta) = p_k(\theta),\ \ k = 1, \ldots, N$$

with $\theta \in \Theta$ as a parameter. Plainly, only parametric functions from $L = \mathrm{span}\{p_1(\theta), p_2(\theta), \ldots, p_N(\theta)\}$ are estimable, i.e., can be unbiasedly estimated. The set $M = \{p_1(\theta), p_2(\theta), \ldots, p_N(\theta)\}$ can be partitioned into

$$M = M_1 \cup \ldots \cup M_r$$

in such a way that

(i) the subspaces $L_1 = \mathrm{span}\,\{M_1\}, \ldots, L_r = \mathrm{span}\,\{M_r\}$ are linearly independent, and

(ii) the partition is maximal, i.e., if for some $l$, $M_l = M_l' \cup M_l''$, $M_l' \cap M_l'' = \emptyset$ and (i) holds for the new partition, then either $M_l = M_l'$ or $M_l = M_l''$.

Such a partition is unique up to ordering. Without loss of generality, one may assume

$$M_1 = \{p_1(\theta), \dots, p_{k_1}(\theta)\}, \quad M_2 = \{p_{k_1+1}(\theta), \dots, p_{k_1+k_2}(\theta)\}, \dots,$$
$$M_r = \left\{p_{k_1+\dots+k_{r-1}+1}(\theta), \dots, p_N(\theta)\right\}. \tag{3}$$

The partition (3) generates a partition of the set $\{1, 2, \dots, N\}$:

$$I_1 \cup I_2 \cup \dots \cup I_r = \{1, \dots, k_1\} \cup \{k_1+1, \dots, k_1+k_2\} \cup \dots \cup \{k_1 + \dots + k_{r-1} + 1, \dots, N\}. \tag{4}$$

A statistic $\widehat{g}(X)$ is a UMVUE if and only if it is constant on the elements of (4),

$$\widehat{g}(x) = const = g_j, \ \ x \in I_j, \ \ j = 1, \dots, r.$$

The subalgebra of UMVUEs is generated by sets (4). The class of parametric functions admitting UMVUEs is a linear subspace of $L$,

$$\text{span } \{\pi_1(\theta), \dots, \pi_r(\theta)\}, \ \ \pi_j(\theta) = \sum_{k \in I_j} p_k(\theta), \quad j = 1, \dots, r. \tag{5}$$

Note that $S(X)$ is the minimal sufficient statistic for $\theta$ if and only if $S(k) = S(l)$ is equivalent to

$$p_k(\theta) = c_{kl} p_l(\theta), \quad \forall \theta \in \Theta \tag{6}$$

for some constant $c_{kl} > 0$. Due to (i), a pair $(k, l) \in \{1, 2, \dots, N\}$ with (6) always belongs to the same element of the partition (4). The above construction is due to Kagan and Konikov (2006). The following example with

$$p_1(\theta) = \theta, \ p_2(\theta) = \theta^2, \ p_3(\theta) = \theta + \theta^2, \ p_4(\theta) = 1 - 2\theta - 2\theta^2, \ \ \theta \in (0, 1/4)$$

illustrates the situation when the minimal sufficient statistic is trivial and incomplete while the subalgebra of UMVUEs is generated by two sets, $\{1, 2, 3\}$ and $\{4\}$. Therefore only elements of span$\{1, \theta + \theta^2\}$ possess UMVUEs.

*2.4. Lehmann's Example.* The following example is due to Lehmann and Scheffé (1950) (see also Lehmann and Casella (1998), pp. 84–85). Let $X \in \{-1, 0, 1, 2, \dots\}$ with

$$P_\theta\,(X = -1) = \theta, \quad P_\theta\,(X = k) = (1 - \theta)^2 \theta^k, \ \ \theta \in (0, 1), \ k = 0, 1, 2, \dots$$

It is easy to see that all unbiased estimators of zero are of the form

$$U(X) = aX \ \text{ for some } \ a \in \mathbb{R}.$$

If $\widehat{g}(X)$ is a UMVUE, then

$$E_\theta \{\widehat{g}(X)X\} = 0, \;\; \theta \in (0,1)$$

so that $X\widehat{g}(X)$ is itself an unbiased estimator of zero. Thus,

$$x\widehat{g}(x) = ax, \;\; x \in \{-1, 0, 1, 2, \ldots\} \;\text{ for some }\; a \in \mathbb{R},$$

whence

$$\widehat{g}(x) = \widehat{g}(-1) \;\text{ for all }\; x \neq 0, \;\text{ with an arbitrary }\; g(0).$$

The subalgebra of UMVUEs in this example is generated by two sets, $\{0\}$ and $\{-1, 1, 2, \ldots\}$. The parametric functions possessing UMVUEs are elements of span$\{1, (1-\theta)^2\}$.

## 3 Properties of MVE-Algebras

In this section some properties of MVE-algebras are presented. They are similar to properties of complete sufficient statistics/subalgebras and, in our opinion, are an argument in favor of that under rather general conditions on a statistical model $(\mathcal{X}, \mathcal{A}, \mathcal{P})$, there exists a maximal MVE-algebra $\widehat{\mathcal{A}}$ such that a statistic $\widehat{g}(X)$ with $E_\theta|\widehat{g}(X)|^2 < \infty, \; \forall\, \theta \in \Theta$ is a UMVUE if and only if it is $\widehat{\mathcal{A}}$-measurable.

Recall that a subalgebra $\widehat{\mathcal{A}} \subset \mathcal{A}$ is an MVE-algebra if any $\widehat{\mathcal{A}}$-measurable statistic $\widehat{g}(X)$ with finite second moment is a UMVUE.

**Theorem 1.** *If $\widetilde{\mathcal{A}}_1, \widehat{\mathcal{A}}_2$ are MVE-algebras, so is $\widehat{\mathcal{A}} = \sigma(\widehat{\mathcal{A}}_1, \widehat{\mathcal{A}}_2)$, the smallest $\sigma$-algebra containing both $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ (in other words, generated by $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$).*

In terms of statistics, if $U_1(X)$, $U_2(X)$ are such that any estimators $\hat{g}_1(U_1(X))$, $\hat{g}_2(U_2(X))$ with finite second moments are UMVUEs, so is $\hat{g}(U_1\;(X), U_2(X))$ with finite second moment.

PROOF. For $A_1 \in \widetilde{\mathcal{A}}_1, A_2 \in \widehat{\mathcal{A}}_2$ set $h_i(x) = 1_{\{x \in A_i\}}, i = 1, 2$. Since $h_1(X)$ is a UMVUE, $\mathrm{cov}_\theta(h_1(X), \chi(X)) = E_\theta(h_1(X), \chi(X)) = 0, \forall\, \theta \in \Theta$ for any $\chi(X)$ with $E_\theta(\chi(X)) = 0, E_\theta|\chi(X)|^2 < \infty, \; \forall\, \theta \in \Theta$. Thus $\chi_1(X) = h_1(X)\chi(X)$ is also a zero mean statistic. Since $|h_1(X)| \leq 1$, $E_\theta|\chi(X)|^2 < \infty$,

$$\mathrm{cov}_\theta\left(h_2(X), \chi_1(X)\right) = E_\theta\left(h_2(X)h_1(X)\chi(X)\right) = 0, \;\; \forall\, \theta \in \Theta$$

and $h_1(X)h_2(X) = 1_{\{X \in A_1 \cap A_2\}}$ is a UMVUE.

Let now $A_{1i} \in \widetilde{\mathcal{A}}_1, A_{2i} \in \widehat{\mathcal{A}}_2, i = 1, \ldots, n$ with indicators $h_{1i}(x) = 1_{\{x \in A_{1i}\}}, h_{2i}(x) = 1_{\{x \in A_{2i}\}}, i = 1, \ldots, n$. The above arguments prove that for any constants $c_1, \ldots, c_n$

$$\sum_{i=1}^{n} c_i h_{1i}(X) h_{2i}(X) \tag{7}$$

is a UMVUE. As is well known, the functions (7) are dense in the Hilbert space $L^2_\theta(\widehat{\mathcal{A}})$ of $\widehat{\mathcal{A}}$-measurable functions $h(X)$ with $\int |h(x)|^2 dP_\theta(x) < \infty$. For an $\widehat{\mathcal{A}}$-measurable statistic $h(X)$ with finite second moment and given $\varepsilon > 0$ take $\widehat{h}(X)$ of the form (7) with $E_\theta |h(X) - \widehat{h}(X)|^2 \leq \varepsilon^2$. Now, for any unbiased estimator of zero $\chi(X)$ with finite second moment,

$$\begin{aligned} |\mathrm{cov}_\theta\,(h(X), \chi(X))\,| &= |\mathrm{cov}_\theta \left(\widehat{h}(X), \chi(X)\right) + \mathrm{cov}_\theta \left(h(X) - \widehat{h}(X), \chi(X)\right)| \\ &\leq \varepsilon \sqrt{E_\theta |\chi(X)|^2}. \end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary, $\mathrm{cov}_\theta\,(h(X), \chi(X)) = 0$ and $h(X)$ is a UMVUE.

The next claim is an immediate corollary of Theorem 2.

**Corollary 1.** *Let $\{\widehat{\mathcal{A}}_\gamma, \gamma \in \Gamma\}$ be the collection of all MVE-algebras indexed by $\gamma \in \Gamma$. Then $\widehat{\mathcal{A}} = \sigma(\widehat{\mathcal{A}}_\gamma,\ \gamma \in \Gamma)$ is the maximal MVE-algebra.*

Turn now to the setup of combining independent data. Let $(\mathcal{X}_i,\ \mathcal{A}_i,\ \mathcal{P}_i)$ be statistical models with $\mathcal{P}_i = \{P_{\theta_i},\ \theta_i \in \Theta_i\}$, $i = 1,\ 2$. Set

$$(\mathcal{X},\ \mathcal{A},\ \mathcal{P}) = (\mathcal{X}_1 \times \mathcal{X}_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2,\ \mathcal{P}_1 \times \mathcal{P}_2)$$

is a statistical model with $\mathcal{P} = \{P_\theta = P_{\theta_1} \times P_{\theta_2}\}$ parameterized by a "bivariate" parameter $\theta = (\theta_1,\ \theta_2) \in \Theta_1 \times \Theta_2 = \Theta$.

**Theorem 2.** *If $\widehat{\mathcal{A}}_i \subset \mathcal{A}_i$ is an MVE-algebra in the model $(\mathcal{X}_i,\ \mathcal{A}_i,\ \mathcal{P}_i)$, $i = 1,\ 2$, then $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_1 \otimes \widehat{\mathcal{A}}_2$ is an MVE-algebra in the model $(\mathcal{X},\ \mathcal{A},\ \mathcal{P})$.*

In terms of statistics, let $X_i \sim P_{\theta_i}$, $i = 1,\ 2$ and $X = (X_1,\ X_2) \sim P_\theta$. If statistics $U_i(X_i)$, $i = 1, 2$ are such that any estimator $\hat{g}_i(U_i(X_i))$ with a finite second moment is a UMVUE of a parametric function $g_i(\theta_i), i = 1,\ 2$, then any estimator $\hat{g}(U_1(X_1),\ U_2(X_2))$ with finite second moment is a UMVUE of some $g(\theta_1, \theta_2)$.

Proof. In view of Theorem 2, suffice to prove that $\widehat{\mathcal{A}}_1$ is an MVE-algebra in the model $(\mathcal{X},\ \mathcal{A},\ \mathcal{P})$. Let $\chi(X_1,\ X_2)$ be a zero mean statistic with finite second moment so that

$$\int_{\mathcal{X}_1} \int_{\mathcal{X}_2} |\chi(x_1,\ x_2)| dP_{\theta_1}(x_1) dP_{\theta_2}(x_2) < \infty,\ (\theta_1,\ \theta_2) \in \Theta.$$

By Fubini theorem, for any $\theta_2$ the function

$$\tilde{\chi}(x_1;\, \theta_2) = \int_{\mathcal{X}_2} \chi(x_1,\, x_2) dP_{\theta_2}(x_2)$$

is well defined for $P_{\theta_1}$-almost all $x_1$ and is a zero mean statistic for the model $(\mathcal{X}_1,\, \mathcal{A}_1,\, \mathcal{P}_1)$. Thus, for any $\widehat{\mathcal{A}}_1$-measurable statistic $U_1(X_1)$ with finite second moment,

$$\begin{aligned} 0 &= \operatorname{cov}(U_1(X_1),\, \tilde{\chi}(X_1;\, \theta_2)) = \int_{\mathcal{X}_1} U_1(x_1)\tilde{\chi}(x_1; \theta_2) dP_{\theta_1}(x_1) \\ &= \int_{\mathcal{X}_1}\int_{\mathcal{X}_2} U_1(x_1)\chi(x_1,\, x_2) dP_{\theta_1}(x_1) dP_{\theta_2} \end{aligned}$$

and $U_1(X_1)$ is a UMVUE in the combined model $(\mathcal{X},\, \mathcal{A},\, \mathcal{P})$.

If in Theorem 3, the condition "$\widehat{\mathcal{A}}_i \subset \mathcal{A}_i$ is an MVE-algebra for $\theta_i$, $i = 1, 2$" is replaced with "$\widehat{\mathcal{A}}_i \subset \mathcal{A}_i$ is a complete sufficient subalgebra", then the claim may be replaced with "$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_1 \otimes \widehat{\mathcal{A}}_2$ is a complete sufficient subalgebra for $(\theta_1,\, \theta_2)$". This result was proved in Landers and Rogge (1976) (where it is stated for arbitrary subalgebras, not necessarily sufficient) strengthening previous results by Plachky (1977) and Fraser (1957) that required some additional conditions on the models.

## 4 Comments

In conclusion we would like to make some general comments on sufficiency and complete sufficiency. While sufficiency plays a fundamental role in all areas of statistical inference, complete sufficiency is tailored for the estimation. In a sense, while sufficiency is a statistical concept, completeness looks more like a mathematical-statistical tool. It seems that a more general concept of an MVE-algebra preserves the basic property of complete sufficiency in some setups when the (minimal) sufficient statistic is incomplete.

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Abram M. Kagan
Department of Mathematics,
University of Maryland,
College Park, MD 20742, USA
E-mail: amk@math.umd.edu

Yaakov Malinovsky
Department of Mathematics
and Statistics, University
of Maryland, Baltimore County,
Baltimore, MD 21250, USA
E-mail: yaakovm@umbc.edu

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